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MATHEMATICS / NDA EXAMINATION

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Sets

Sets, counting and Venn diagrams

01 Count a union without double counting

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

Finite sets; do not count the overlap twice.

02 Take complements using De Morgan's laws

$$(A \cup B)' = A' \cap B', \quad (A \cap B)' = A' \cup B'$$

Complements use the same universal set.

03 Count subsets and Cartesian-product pairs

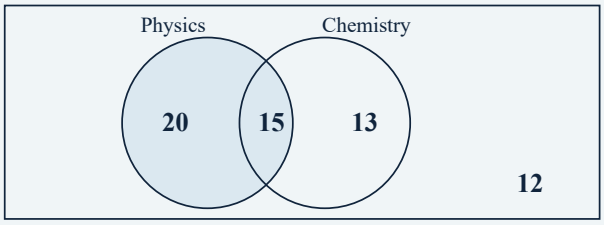
$$n(\mathcal{P}(A)) = 2^m, \quad n(A \times B) = n(A)n(B)$$

A has m elements; the power set includes the empty set.

Quick method. For at most half the elements of an odd-sized set, pair each subset with its complement. Exactly one member of each pair is counted.

EXAMPLE 1 **Of 60 students, 35 study Physics, 28 Chemistry and 15 both. How many study neither ?**

Find the union first: 35 + 28 - 15 = 48. Subtract from the universal total.



Overlap is counted in both totals.

Subtract it once for the union.

The outside region gives neither.

$$60 - (35 + 28 - 15) = 12$$

EXAMPLE 2 **A set has 9 elements. How many subsets have at most 4 elements ?**

Complement-pair sizes add to 9, so each pair has exactly one subset of size at most 4. Take half of all subsets.

$$2^9/2 = 256$$

Watch for. The empty set is a subset of every set; it need not be an element of every set.

Relations and Functions

Relations and equivalence classes

01 Count relations between two finite sets

$$R \subseteq A \times B, \quad \#\{R \subseteq A \times B\} = 2^{mn}$$

Here m = n(A), n = n(B); this counts all relations, not functions.

02 Recognise an equivalence relation

$$aRa; \quad aRb \Rightarrow bRa; \quad aRb,\ bRc \Rightarrow aRc$$

Reflexive, symmetric and transitive: all three define equivalence.

03 Identify an equivalence class

$$[a] = \{x \in A : xRa\}$$

Equivalence classes partition the underlying set.

Quick method. Test the three properties separately. One counterexample is enough to disprove a property; a few successful pairs cannot prove it.

EXAMPLE 1 A has 2 elements and B has 3. How many relations from A to B are possible ?

There are 6 ordered pairs. A relation chooses any subset of these pairs.

$$2^{2 \cdot 3} = 64$$

EXAMPLE 2 On the integers, a is related to b when a - b is divisible by 4. How many equivalence classes arise ?

Equality of remainder modulo 4 is reflexive, symmetric and transitive. The possible remainders identify the classes.

$$[0],\ [1],\ [2],\ [3] \quad \Rightarrow \quad 4$$

Watch for. A symmetric relation need not be transitive. A relation may be empty without being reflexive on a nonempty set.

Relations and Functions

Functions, domain, composition and inverses

01 Compose functions in the correct order

$$(f \circ g)(x) = f(g(x)), \quad f^{-1}(f(x)) = x$$

An inverse requires f to be bijective on the declared domain and codomain.

02 Count functions and injective maps

$$\#\{f : A \to B\} = n^m, \quad \#\text{injective maps} = \frac{n!}{(n-m)!}$$

Finite sets of sizes m,n; injection count assumes m <= n.

03 Recognise even and odd functions

$$f(-x) = f(x) \text{ (even)}, \quad f(-x) = -f(x) \text{ (odd)}$$

The domain must be symmetric about zero for these definitions.

Quick method. Work inside-out in composition and enforce both domains. If f(a) = a, every repeated composition at a stays a.

EXAMPLE 1 **Find the domain of square root of (x - 1) divided by (x - 3).**

The radicand needs x >= 1; the denominator excludes x = 3.

$$D = [1, 3) \cup (3, \infty)$$

EXAMPLE 2 **If f(x) = (2x + 1)/(x + 2), find f inverse of x.**

Set y = f(x), rearrange to x(y - 2) = 1 - 2y, then interchange variables.

$$f^{-1}(x) = \frac{1-2x}{x-2} \quad (x \neq 2)$$

Watch for. An inverse function and a reciprocal are different: f inverse is not generally 1/f.

Algebra

Binary numbers, indices and logarithms

01 Read binary place values

$$(b_n b_{n-1} \ldots b_0)_2 = \sum_{k=0}^{n} b_k 2^k$$

Every binary digit is 0 or 1.

02 Combine and change logarithm bases

$$\log_a (xy) = \log_a x + \log_a y, \quad \log_a x = \frac{\ln x}{\ln a}$$

a > 0, a not 1, and logarithm arguments positive.

03 Use exponent laws consistently

$$a^{p+q} = a^p a^q, \quad (a^p)^q = a^{pq}$$

For arbitrary real powers, use a positive base.

Quick method. For binary arithmetic, convert small values to decimal, calculate, then convert back. For logarithmic equations, check the original domain after solving.

EXAMPLE 1 Find the LCM of binary 10100 and 11110, and write it in binary.

Convert to 20 and 30. Their LCM is 60. Expand 60 using powers of 2.

$$60 = 32 + 16 + 8 + 4 = (111100)_2$$

EXAMPLE 2 Solve log base 2 of (x - 1) plus log base 2 of (x + 1) = 3.

The domain is x > 1. Combine logarithms: x^2 - 1 = 8; discard the negative root.

$$x^2 = 9, \quad x > 1 \quad \Rightarrow \quad x = 3$$

Watch for. Logarithms do not distribute over addition: log(x + y) is not log x + log y.

Algebra

Quadratics: roots, signs and parameters

01 Use root sums, products and discriminant

$$\alpha+\beta=-\frac{b}{a},\quad \alpha\beta=\frac{c}{a},\quad D=b^2-4ac$$

For $ax^2 + bx + c = 0$ with a not zero.

02 Form symmetric expressions in roots

$$\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta$$

Use symmetric expressions before finding individual roots.

03 Check positivity for every real input

$$a>0,\ D<0\Rightarrow ax^2+bx+c>0 \text{ for all real } x$$

For nonnegative values at every real x, replace D < 0 by D <= 0.

Quick method. Two positive real roots need D >= 0, positive sum and positive product. All three conditions matter.

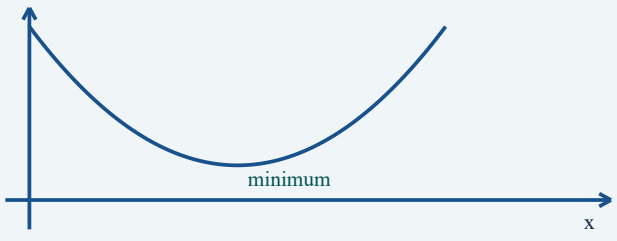
EXAMPLE 1 If the roots of $2x^2 - 5x + 2$ are alpha and beta, find alpha squared + beta squared.

Use sum = 5/2 and product = 1; do not solve the quadratic first.

$$\alpha^2+\beta^2=\frac{25}{4}-2=\frac{17}{4}$$

EXAMPLE 2 Find all real k for which $x^2 + kx + 4$ is positive for every real x.

The leading coefficient is positive. Strict positivity excludes a real zero, so its discriminant must be negative.



An upward-opening quadratic has a minimum.

Complete the square to locate the vertex.

Schematic: vertical position is illustrative.

$$k^2-16<0\Rightarrow -4<k<4$$

Watch for. Squaring roots can change the root set. Also distinguish a repeated real root from no real roots.

Algebra

Inequalities and graphical feasible regions

01 Translate an absolute-value inequality

$$|x| < a \Leftrightarrow -a < x < a \quad (a > 0)$$

For |x| > a, use the union of two outside intervals.

02 Use the arithmetic-geometric mean bound

$$u + v \geq 2\sqrt{uv} \quad (u, v \geq 0)$$

Equality in AM-GM requires u = v.

03 Intersect graphical half-planes

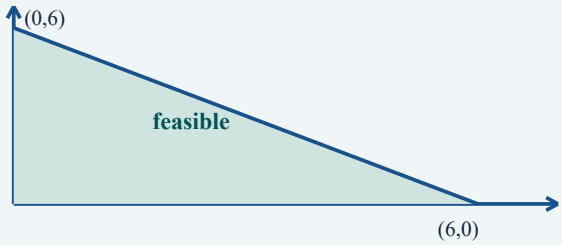
$$ax + by \leq c$$

Plot the boundary line, test a point, then intersect the required half-planes.

Quick method. For a linear objective on a bounded polygon, check its vertices. For rational inequalities, mark zeros and excluded denominator values before a sign chart.

EXAMPLE 1 **Maximise x + 2y when x >= 0, y >= 0 and x + y <= 6.**

The feasible triangle has vertices (0,0), (6,0), (0,6). Compare the objective there.



Intersect all required half-planes.

For this bounded triangle, test all vertices.

The objective x + 2y peaks at (0,6).

$$\max(0, 6, 12) = 12 \quad \text{at } (0, 6)$$

EXAMPLE 2 **Solve (x - 1)/(x + 2) > 0.**

Critical values are -2 and 1. Numerator and denominator have the same sign on the two outside intervals.

$$x \in (-\infty, -2) \cup (1, \infty)$$

Watch for. Multiplying an inequality by a quantity of unknown sign can reverse its direction. Strict boundaries are excluded.

Complex Numbers

Complex numbers: modulus and geometry

01 Separate real part, conjugate and modulus

$$z = a + ib, \quad \bar{z} = a - ib, \quad |z| = \sqrt{a^2 + b^2}$$

The principal argument is commonly taken in (-pi, pi].

02 Use the conjugate for division

$$z\bar{z} = |z|^2, \quad \frac{1}{z} = \frac{\bar{z}}{|z|^2}$$

The reciprocal requires z not zero.

03 Apply the triangle inequality

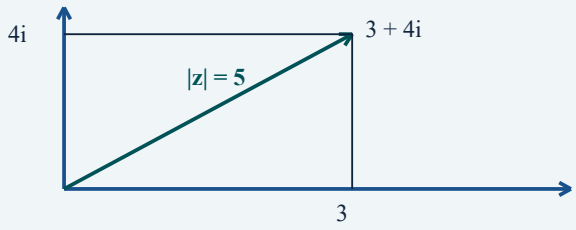
$$|z_1 + z_2| \le |z_1| + |z_2|$$

Equality holds when nonzero numbers have the same direction.

Quick method. Interpret |z - a| as distance from a in the Argand plane. A distance equation often identifies a line or circle faster than expansion.

EXAMPLE 1 **Simplify (3 + 4i)/(1 - 2i).**

Multiply numerator and denominator by 1 + 2i. The denominator becomes 5.



Real and imaginary parts are coordinates.

Modulus is distance from the origin.

Argument is measured from the positive real axis.

$$\frac{3+4i}{1-2i} = \frac{-5+10i}{5} = -1 + 2i$$

EXAMPLE 2 **Find the locus of z = x + iy satisfying |z - 2| = |z + 2|.**

The point is equidistant from 2 and -2 on the real axis. Use their perpendicular bisector.

$$(x-2)^2 + y^2 = (x+2)^2 + y^2 \Rightarrow x = 0$$

Watch for. The argument of a product is the sum only modulo 2π; restore it to the chosen principal range.

Complex Numbers

Polar form and roots of unity

01 Raise a complex number in polar form

$$z = r(\cos\theta + i\sin\theta), \quad z^n = r^n(\cos n\theta + i\sin n\theta)$$

De Moivre's formula for integer n; negative powers require z not zero, and 0 to the power 0 is excluded.

02 List all nth roots

$$z_k = r^{1/n}(\cos\theta_k + i\sin\theta_k)$$
$$\theta_k = (\theta + 2k\pi)/n$$

For a positive integer n and nonzero z (r > 0), use k = 0, ..., n - 1.

03 Use the nonreal cube roots of unity

$$\omega^3 = 1, \quad 1 + \omega + \omega^2 = 0, \quad \omega \neq 1$$

The non-real cube roots are omega and omega squared.

Quick method. Reduce powers of omega modulo 3 and powers of i modulo 4. The roots of a nonzero complex number are equally spaced on a circle.

EXAMPLE 1 **Evaluate (1 + i) to the eighth power.**

Its modulus is square root of 2 and its argument is π/4. Raise the modulus and multiply the argument.

$$(1+i)^8 = (\sqrt{2})^8(\cos 2\pi + i\sin 2\pi) = 16$$

EXAMPLE 2 **For a non-real cube root omega of unity, evaluate (1 - omega)(1 - omega squared).**

Expand, then use omega + omega squared = -1 and omega cubed = 1.

$$1 - (\omega + \omega^2) + \omega^3 = 3$$

Watch for. There are n distinct nth roots of a nonzero number, but zero has only one distinct nth root.

Sequences and Series

Arithmetic, geometric and harmonic progressions

01 Identify an arithmetic progression

$$a_n = a + (n-1)d, \quad S_n = \frac{n}{2}[2a + (n-1)d]$$

Arithmetic progression: a is the first term and d the common difference.

02 Sum geometric progressions

$$S_n = \frac{a(1-r^n)}{1-r}, \quad S_\infty = \frac{a}{1-r}$$

Finite formula: r not 1; infinite sum: |r| < 1.

03 Compare arithmetic, geometric and harmonic means

$$H = \frac{2ab}{a+b}, \quad A \geq G \geq H$$

For two positive numbers; reciprocals of an HP form an AP.

Quick method. Use the term formula for a position and the sum formula for a total. In a harmonic progression, work with reciprocals first.

EXAMPLE 1 **An AP has third term 11 and eighth term 31. Find the sum of its first ten terms.**

Subtract the equations: 5d = 20, hence d = 4 and a = 3. Then use the AP sum.

$$S_{10} = \frac{10}{2}[6 + 9(4)] = 210$$

EXAMPLE 2 **Find the harmonic mean of 3 and 6.**

Average the reciprocals, then invert. This is equivalent to the two-number HM formula.

$$H = \frac{2(3)(6)}{3+6} = 4$$

Watch for. An infinite GP does not converge merely because r < 1: the condition is |r| < 1.

Permutations and Combinations

Permutations and combinations

01 Decide whether order matters

$$^{n}P_{r} = \frac{n!}{(n-r)!}, \quad \binom{n}{r} = \frac{n!}{r!\,(n-r)!}$$

Use permutations for order; combinations for selection. Here 0 <= r <= n.

02 Allow for repeated or circular arrangements

$$\#\text{arrangements} = \frac{n!}{p!\,q!\,\ldots}, \quad \#\text{circular} = (n-1)!$$

Repeated identical objects; circular count assumes rotations identical but reflections distinct.

03 Compare adjacent binomial coefficients

$$\frac{\binom{n}{r+1}}{\binom{n}{r}} = \frac{n-r}{r+1}$$

Adjacent-binomial ratios often avoid large factorials.

Quick method. For at least one, count all minus none. For people who must sit together, treat the group as one block and arrange inside it.

EXAMPLE 1 **How many distinct arrangements of the letters of LEVEL are possible ?**

There are 5 letters, with L repeated twice and E twice. Divide out the repeated arrangements.

$$\frac{5!}{2!\,2!} = 30$$

EXAMPLE 2 **Find the greatest integer r for which C(12,r+1) > 2C(12,r), with 0 <= r <= 11.**

Divide by the positive denominator. Solve (12 - r)/(r + 1) > 2, giving r < 10/3.

$$r_{\max} = 3$$

Watch for. A leading zero is forbidden in a multi-digit number. Different letters/people are distinct unless the question says otherwise.

Binomial Theorem

Binomial theorem and coefficient extraction

01 Expand a nonnegative integer power

$$(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^{n-r} b^r$$

n is a nonnegative integer in this finite expansion.

02 Locate the required general term

$$T_{r+1} = \binom{n}{r} a^{n-r} b^r$$

The term number is r + 1, not r.

03 Sum or alternate the binomial coefficients

$$\sum_r \binom{n}{r} = 2^n, \qquad \sum_r (-1)^r \binom{n}{r} = 0$$

The alternating identity needs n >= 1.

Quick method. For a constant term, set the exponent of x to zero. Rewrite reciprocal factors before multiplying two long expansions.

EXAMPLE 1 **Find the constant term in $(x^2 + 1/x)$ to the sixth power.**

The exponent in the general term is 12 - 3r. Set it to zero, giving r = 4.

$$\binom{6}{4} = 15$$

EXAMPLE 2 **Find the coefficient of x^3 in (1 + x) cubed times (1 + 1/x) cubed.**

Rewrite the product as x to the power -3 times (1 + x) to the sixth. Now select x to the sixth.

$$[x^3]\, x^{-3}(1+x)^6 = \binom{6}{6} = 1$$

Watch for. The middle term depends on whether n + 1, the number of terms, is odd or even.

Matrices and Determinants

Matrix operations and special matrices

01 Reverse the order when transposing a product

$$(AB)^T = B^T A^T, \quad (A+B)^T = A^T + B^T$$

The product AB requires columns(A) = rows(B).

02 Recognise symmetric and skew-symmetric matrices

$$A^T = A \text{ (symmetric)}, \quad A^T = -A \text{ (skew-symmetric)}$$

Over the real numbers, skew-symmetric matrices have zero diagonal.

03 Split a matrix into symmetric parts

$$A = \frac{A + A^T}{2} + \frac{A - A^T}{2}$$

Every real square matrix splits into symmetric and skew-symmetric parts.

Quick method. Check dimensions before multiplying. For powers, use a relation such as A squared = A or A squared = I instead of repeated multiplication.

EXAMPLE 1 **A is a 2 by 3 matrix and B is a 3 by 4 matrix. What is the order of AB ?**

The inner dimensions agree. The product keeps the two outer dimensions.

$$(2 \times 3)(3 \times 4) \Rightarrow 2 \times 4$$

EXAMPLE 2 **If A squared = A, simplify (I + A) cubed.**

Use the binomial expansion because I commutes with A, then replace all positive powers of A by A.

$$(I + A)^3 = I + 3A + 3A + A = I + 7A$$

Watch for. In general AB is not BA; AB = 0 does not imply A = 0 or B = 0.

Matrices and Determinants

Determinants, adjoints and inverses

01 Track determinant scaling and products

$$\det(AB) = \det A \det B, \quad \det(kA) = k^n \det A$$

A and B are square of order n.

02 Connect adjoint, determinant and inverse

$$A\,\mathrm{adj}(A) = \det(A) I, \quad A^{-1} = \frac{\mathrm{adj}(A)}{\det A}$$

An inverse exists exactly when det(A) is nonzero.

03 Reverse the order when inverting a product

$$(AB)^{-1} = B^{-1}A^{-1}, \quad \det(A^{-1}) = \frac{1}{\det A}$$

Both matrices must be invertible.

Quick method. Adding a multiple of one row to another leaves the determinant unchanged. Use this to create zeros before expanding.

EXAMPLE 1 **For a 3 by 3 matrix A with determinant 2, find det(3A) and det(adj A).**

Scaling every row gives a factor 3 cubed. For the adjoint use det(adj A) = det(A) to the power n - 1.

$$\det(3A) = 54, \quad \det(\mathrm{adj}A) = 4$$

EXAMPLE 2 **Find the inverse of the diagonal matrix with diagonal entries 2 and -3.**

Invert each nonzero diagonal entry; the off-diagonal entries stay zero.

$$A = \begin{bmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{bmatrix} \quad \text{adj } A = \begin{bmatrix} \mathbf{d} & \mathbf{-b} \\ \mathbf{-c} & \mathbf{a} \end{bmatrix}$$

Swap the diagonal entries; negate the others.

Divide every adjoint entry by ad - bc.

The determinant must be nonzero.

$$A^{-1} = \mathrm{diag}\left(\tfrac{1}{2}, -\tfrac{1}{3}\right)$$

Watch for. Swapping two rows reverses the determinant sign; multiplying one row by k multiplies it by k, not k to the nth power.

Matrices and Determinants

Linear systems: Cramer and matrix methods

01 Write the system as a matrix equation

$$AX = B, \quad X = A^{-1}B$$

The inverse method requires a nonsingular coefficient matrix.

02 Replace columns in Cramer's rule

$$x_i = \frac{\Delta_i}{\Delta} \quad (\Delta \neq 0)$$

Replace column i by the constants to form Delta i.

03 Test the ranks for consistency

$$\operatorname{rank}(A) = \operatorname{rank}([A|B])$$

Consistency condition; uniqueness additionally needs rank equal to the number of unknowns.

Quick method. For small systems, elimination is often faster than constructing the full inverse. A zero determinant means inspect consistency, not automatically no solution.

EXAMPLE 1 **Solve 2x + y = 5 and x - y = 1.**

Add the equations to get 3x = 6. Substitute x = 2 into either equation.

$$x = 2, \quad y = 1$$

EXAMPLE 2 **Classify x + y = 2 and 2x + 2y = 5.**

Twice the first left side equals the second left side, but twice its constant is 4, not 5.

$$\operatorname{rank}(A) = 1 < 2 = \operatorname{rank}([A|B])$$

Watch for. Proportional coefficient rows with unequal constant ratios give no solution; proportional full rows can give infinitely many solutions.

Trigonometry

Angle measures and core identities

01 Use radians for arc length and sector area

$$\pi \text{ rad} = 180^\circ, \quad s = r\theta, \quad A_{\text{sector}} = \tfrac{1}{2}r^2\theta$$

Arc and sector formulas require theta in radians.

02 Apply the Pythagorean identities

$$\sin^2\theta + \cos^2\theta = 1, \quad 1 + \tan^2\theta = \sec^2\theta$$

Use identities only where both sides are defined.

03 Recover signs from the quadrant

$$\sin(\pi - \theta) = \sin\theta, \quad \cos(\pi - \theta) = -\cos\theta$$

The quadrant determines the signs.

Quick method. Reduce an angle to a familiar reference angle, then attach the quadrant sign. Keep exact surd values instead of early decimals.

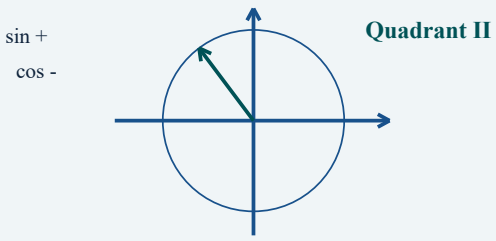
EXAMPLE 1 **Find the area of a sector of radius 6 with central angle 60 degrees.**

Convert 60 degrees to pi/3 radians, then apply half r squared theta.

$$A = \tfrac{1}{2}(36)\tfrac{\pi}{3} = 6\pi$$

EXAMPLE 2 **If sin theta = 3/5 and theta lies in the second quadrant, find tan theta.**

Cosine is negative in quadrant II. Its magnitude is 4/5; divide sine by cosine.



The unit-circle point is (cos theta, sin theta).

Signs depend on the quadrant.

One complete turn is 2pi radians.

$$\cos\theta = -\tfrac{4}{5}, \quad \tan\theta = -\tfrac{3}{4}$$

Watch for. A degree-mode value cannot be inserted into a radian-based limit or derivative formula without conversion.

Trigonometry

Compound angles, multiples and transformations

01 Expand a sine of two angles

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

Track the sign attached to B.

02 Transform double-angle expressions

$$\cos 2A = 1 - 2\sin^2 A = 2\cos^2 A - 1, \quad \sin 2A = 2\sin A \cos A$$

These forms reduce squared trigonometric expressions.

03 Convert a sum into a product

$$\sin A + \sin B = 2\sin \frac{A+B}{2} \cos \frac{A-B}{2}$$

Product-to-sum is the reverse transformation.

04 Use triple-angle and half-angle forms

$$\sin 3A = 3\sin A - 4\sin^3 A, \quad \tan \frac{A}{2} = \frac{\sin A}{1+\cos A}$$

The half-angle quotient requires a nonzero denominator.

Quick method. For a sin x + b cos x, combine as R sin(x + phi), with R = square root of (a squared + b squared). This exposes the range.

EXAMPLE 1 **Evaluate sin 75 degrees exactly.**

Write 75 as 45 + 30 and apply the compound-angle identity.

$$\sin 75^\circ = \frac{\sqrt{6}+\sqrt{2}}{4}$$

EXAMPLE 2 **Find the maximum value of 3 sin x + 4 cos x for real x.**

The amplitude is square root of 9 + 16. The phase-shifted sine can attain 1.

$$-5 \le 3\sin x + 4\cos x \le 5$$

Watch for. Dividing by sin x or cos x can remove legitimate zero cases. Check them separately first.

Trigonometry

Trigonometric equations and inverse branches

01 List every solution of a sine equation

$$\sin x = \sin a \Rightarrow x = n\pi + (-1)^n a$$

n is any integer.

02 Use the period of tangent

$$\tan x = \tan a \Rightarrow x = n\pi + a$$

Exclude points at which tangent is undefined.

03 Respect the principal inverse ranges

$$\sin^{-1} x \in [-\pi/2, \pi/2], \quad \cos^{-1} x \in [0, \pi]$$

Their real input domains are [-1,1]. Arctan has range (-π/2, π/2).

04 Relate inverse sine and inverse cosine

$$\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$$

Valid for -1 <= x <= 1 on principal branches.

Quick method. An inverse trigonometric answer must lie in its principal interval. Reduce the original angle before cancelling an inverse with a function.

EXAMPLE 1 Solve sin x = 1/2 on 0 <= x < 2π.

The reference angle is π/6. Sine is positive in quadrants I and II.

$$x = \frac{\pi}{6}, \ \frac{5\pi}{6}$$

EXAMPLE 2 Evaluate the principal value of arcsin(sin(5π/6)).

5π/6 is outside the arcsine range. The angle π/6 has the same sine and is inside that range.

$$\sin^{-1}\left(\sin\frac{5\pi}{6}\right) = \frac{\pi}{6}$$

Watch for. Sin inverse of sin x is not always x; it is the principal angle with the same sine.

Trigonometry

Triangles, heights and distances

01 Connect sides, angles and circumradius

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

Sides are opposite the corresponding capital-letter angles.

02 Apply the cosine rule and sine area

$$c^2 = a^2 + b^2 - 2ab\cos C, \quad \Delta = \tfrac{1}{2}ab\sin C$$

C is the included angle between a and b.

03 Use semiperimeter, inradius and circumradius

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)} = rs, \quad R = \frac{abc}{4\Delta}$$

s = (a + b + c)/2; r and R are inradius and circumradius.

Quick method. Draw the right triangle for elevation problems. With two observation points, write the same height in both tangent equations before solving distances.

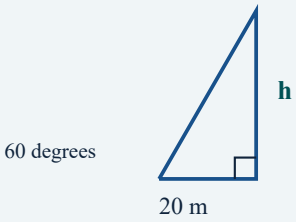
EXAMPLE 1 **A triangle has sides 13, 14 and 15. Find its area and inradius.**

The semiperimeter is 21. Heron's formula gives area 84; divide by 21 for the inradius.

$$\Delta = \sqrt{21 \cdot 8 \cdot 7 \cdot 6} = 84, \quad r = 4$$

EXAMPLE 2 **From a point 20 m from a tower's foot, its top has elevation 60 degrees. Find the height.**

Use opposite = adjacent times tangent in the right triangle. Assume level ground and observation from ground level.



tan(theta) = height / horizontal distance.

The right angle belongs at the tower foot.

Schematic; angles and lengths are not to scale.

$$h = 20\tan 60^\circ = 20\sqrt{3} \text{ m}$$

Watch for. An angle of depression equals the corresponding elevation only because the horizontal lines are parallel.

Geometry

Straight lines, distances and section formula

01 Find distance and slope

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}, \quad m = \frac{y_2 - y_1}{x_2 - x_1}$$

Slope form excludes vertical lines; their equation is x = constant.

02 Measure a perpendicular point-line distance

$$d(P, ax + by + c = 0) = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}$$

The coefficients a and b are not both zero.

03 Divide a line segment internally

$$P = \left(\frac{mx_2 + nx_1}{m + n}, \frac{my_2 + ny_1}{m + n}\right)$$

Internal division AP:PB = m:n. For external division use signed ratios.

04 Calculate the angle between two lines

$$\tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right|$$

Finite slopes and 1 + m1*m2 not zero. If m1*m2 = -1, the lines are perpendicular and theta = 90 degrees.

Quick method. For area or collinearity, use a coordinate determinant. For parallel/perpendicular lines, compare direction or normal vectors rather than forcing a slope.

EXAMPLE 1 Find the perpendicular distance of (1,2) from 3x + 4y - 6 = 0.

Substitute into the numerator and divide by the normal's length.

$$d = \frac{|3 + 8 - 6|}{5} = 1$$

EXAMPLE 2 Find the point dividing A(1,2), B(7,8) internally in ratio 1:2.

Weight B by 1 and A by 2; divide by 3.

$$P = \left(\frac{7 + 2}{3}, \frac{8 + 4}{3}\right) = (3, 4)$$

Watch for. The coefficients of a line give a normal vector (a,b), not a direction vector; (-b,a) is a direction.

Geometry

Circles, tangents and chord geometry

01 Read a circle in general form

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

Centre (-g,-f); radius squared = g squared + f squared - c.

02 Read a circle in centre-radius form

$$(x - h)^2 + (y - k)^2 = r^2$$

A line is tangent when its distance from the centre equals r.

03 Write a tangent at a point of contact

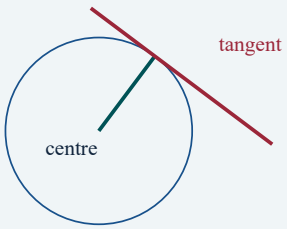
$$xx_1 + yy_1 = r^2$$

Tangent at (x1,y1) on the circle centred at the origin.

Quick method. Complete squares before reading the centre. A point outside a circle has tangent length square root of (distance-to-centre squared minus radius squared).

EXAMPLE 1 Find the centre and radius of $x^2 + y^2 - 6x + 4y - 12 = 0$.

Complete the squares: add 9 and 4 to the constant side.



The radius to the contact point is normal to the tangent. Distance from the centre to a tangent equals the radius.

$$(x - 3)^2 + (y + 2)^2 = 25 \Rightarrow C = (3, -2), \quad r = 5$$

EXAMPLE 2 Find the tangent to $x^2 + y^2 = 25$ at (3,4).

The point lies on the circle. Substitute its coordinates into xx1 + yy1 = r squared.

$$3x + 4y = 25$$

Watch for. A tangent formula at a point assumes that point lies on the circle. A negative radius squared gives no real circle.

Geometry

Parabola: focus, directrix and tangent

01 Locate the focus and directrix

$$y^2 = 4ax: \quad F = (a, 0), \quad \text{directrix } x = -a$$

For a > 0 the parabola opens to the right; latus rectum length is 4a.

02 Use a parametric point and tangent

$$P = (at^2, 2at), \quad ty = x + at^2$$

Parametric point and its tangent for $y^2 = 4ax$.

03 Write the tangent at a known point

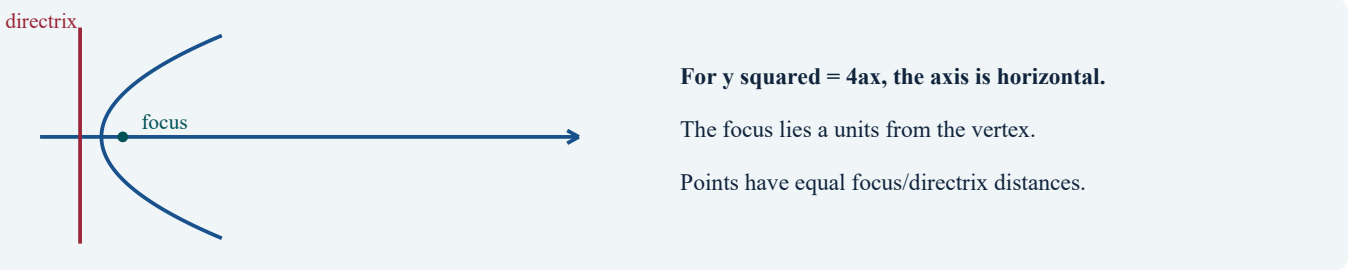
$$yy_1 = 2a(x + x_1)$$

Tangent at a point (x1,y1) on the parabola.

Quick method. Match the squared coordinate first: y^2 means a horizontal axis, x^2 a vertical axis. The unsquared coefficient sets a and the opening direction.

EXAMPLE 1 **Find the focus and directrix of $y^2 = 12x$.**

Compare with $y^2 = 4ax$, giving a = 3.



$$F = (3, 0), \quad x = -3$$

EXAMPLE 2 **Find the tangent to $y^2 = 8x$ at parameter t = 2.**

Here a = 2. Substitute t = 2 into $ty = x + at^2$.

$$2y = x + 8$$

Watch for. The coefficient of x is 4a, not a. Do not interchange focus distance a with latus rectum length 4a.

Geometry

Ellipse and hyperbola: standard forms

01 Recognise the standard ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \quad c^2=a^2-b^2, \quad e=\frac{c}{a}$$

Ellipse with a > b > 0; foci (+/-c,0), 0 < e < 1.

02 Recognise the standard hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1, \quad c^2=a^2+b^2$$

Hyperbola: foci (+/-c,0), e > 1; asymptotes y = +/-bx/a.

03 Choose the correct conic tangent sign

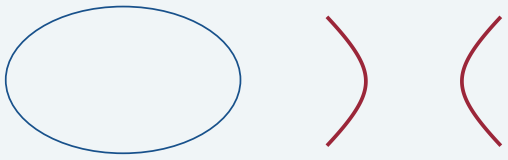
$$\frac{xx_1}{a^2}\pm\frac{yy_1}{b^2}=1$$

Tangent at a point on the corresponding ellipse (+) or hyperbola (-).

Quick method. For an ellipse, the larger denominator identifies the major axis. For a hyperbola, the positive squared term identifies the transverse axis.

EXAMPLE 1 **Find the eccentricity and foci of $x^2/25 + y^2/9 = 1$.**

Use a = 5, b = 3; c squared = 25 - 9 = 16.



ellipse: e < 1 hyperbola: e > 1

Ellipse: sum of focal distances is constant.

Hyperbola: absolute difference is constant.

The parabola has eccentricity 1.

$$c=4, \quad e=\frac{4}{5}, \quad F=(\pm 4, 0)$$

EXAMPLE 2 **Find the asymptotes of $x^2/9 - y^2/16 = 1$.**

Here a = 3 and b = 4. Use y = +/-bx/a.

$$y=\pm\frac{4}{3}x$$

Watch for. Ellipse and hyperbola use different signs in c squared. In a hyperbola a need not be larger than b.

3-Dimensional Geometry

3D points, directions and lines

01 Measure distance in three dimensions

$$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2$$

The 2D distance formula extends by one more squared difference.

02 Normalise direction ratios

$$l^2 + m^2 + n^2 = 1, \quad (l, m, n) = \frac{(a, b, c)}{\sqrt{a^2 + b^2 + c^2}}$$

Normalise nonzero direction ratios; a,b,c must not all be zero.

03 Describe and compare spatial lines

$$\mathbf{r} = \mathbf{a} + t\mathbf{b}, \quad \cos\theta = \frac{|\mathbf{b}_1 \cdot \mathbf{b}_2|}{|\mathbf{b}_1||\mathbf{b}_2|}$$

Both directions are nonzero; absolute value selects the smaller angle, from 0 to 90 degrees.

Quick method. Use vector form when a direction ratio is zero. Find intersections by matching all three coordinates, not only two.

EXAMPLE 1 **Find the distance between (1,2,3) and (3,5,9).**

The coordinate differences are 2, 3 and 6. Square, add and take the square root.

$$d = \sqrt{4 + 9 + 36} = 7$$

EXAMPLE 2 **Find the direction cosines of a direction with ratios 2,-1,2.**

Its magnitude is 3. Divide each component by 3.

$$(l, m, n) = \left(\frac{2}{3}, -\frac{1}{3}, \frac{2}{3}\right)$$

Watch for. Two nonparallel lines in 3D may be skew and have no intersection; this cannot happen for two nonparallel coplanar lines.

3-Dimensional Geometry

Planes and spheres

01 Use a plane normal for perpendicular distance

$$ax + by + cz + d = 0, \quad D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

A plane's normal is (a,b,c), which is nonzero.

02 Find the angle between a line and a plane

$$\sin\phi = \frac{|\mathbf{b} \cdot \mathbf{n}|}{|\mathbf{b}||\mathbf{n}|}$$

Smaller line-plane angle, from 0 to 90 degrees; direction b and normal n must both be nonzero.

03 Read a sphere's centre and radius

$$(x - h)^2 + (y - k)^2 + (z - l)^2 = R^2$$

Sphere with centre (h,k,l); plane-plane angles use their normal vectors.

Quick method. For line-plane intersection, substitute the parametric line into the plane. For a sphere, complete three squares before identifying the radius.

EXAMPLE 1 **Find the distance of (1,1,1) from x + 2y + 2z - 9 = 0.**

The numerator is |1 + 2 + 2 - 9| = 4 and the normal length is 3.

$$D = \frac{4}{3}$$

EXAMPLE 2 **Find the sphere whose diameter endpoints are (0,0,0) and (2,4,4).**

The centre is their midpoint (1,2,2). Half the endpoint distance is 3.

$$(x - 1)^2 + (y - 2)^2 + (z - 2)^2 = 9$$

Watch for. The angle with a plane is complementary to the angle with its normal; use sine for line-plane and cosine for normal-normal.

Limits and Continuity

Limits and standard approximations

01 Use the small-angle trigonometric limits

$$\lim_{x\to 0}\frac{\sin x}{x}=1, \quad \lim_{x\to 0}\frac{1-\cos x}{x^2}=\frac{1}{2}$$

Angles are in radians.

02 Use exponential and logarithmic limits

$$\lim_{x\to 0}\frac{e^x-1}{x}=1, \quad \lim_{x\to 0}\frac{\ln(1+x)}{x}=1$$

The logarithm is natural.

03 Keep the first-order binomial term

$$(1+x)^a=1+ax+o(x) \quad (x\to 0)$$

a is fixed; use enough terms when first-order terms cancel.

Quick method. Try factorisation, conjugates and standard-limit substitution before a longer method. A two-sided limit needs the same value from both sides.

EXAMPLE 1 **Evaluate the limit of sin(5x)/x as x tends to zero.**

Factor out 5 and use sin(5x)/(5x) tending to 1.

$$\lim_{x\to 0}\frac{\sin 5x}{x}=5$$

EXAMPLE 2 **Evaluate the limit of (square root of (1 + x) - 1)/x as x tends to zero.**

Multiply by the conjugate. Cancel x, then substitute zero into the remaining denominator.

$$\lim_{x\to 0}\frac{1}{\sqrt{1+x}+1}=\frac{1}{2}$$

Watch for. Direct substitution giving 0/0 is an indeterminate form, not an answer of zero.

Limits and Continuity

Continuity and differentiability

01 Match the value and both one-sided limits

$$\lim_{x\to a^-} f(x) = f(a) = \lim_{x\to a^+} f(x)$$

Continuity at an interior point requires all three quantities to agree.

02 Use the derivative definition

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

A finite derivative gives continuity, but continuity need not give a derivative.

03 Handle a corner in an absolute-value graph

$$\frac{d}{dx}|x-a| = -1 \ (x<a), \qquad +1 \ (x>a)$$

The derivative is undefined at x = a.

Quick method. For piecewise functions, match values first and one-sided slopes second. For sums of absolute values, split at every breakpoint.

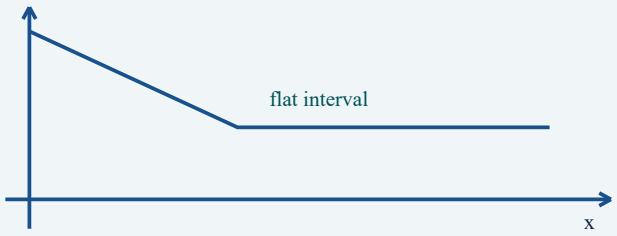
EXAMPLE 1 Choose k so that f(x) = $(x^2 - 1)/(x - 1)$ for x not 1 and f(1) = k is continuous.

Cancel the factor away from 1; its limiting expression is x + 1.

$$k = \lim_{x\to 1} (x+1) = 2$$

EXAMPLE 2 For f(x) = |x - 2| + |x - 5|, find the derivative at x = 3.

Between 2 and 5, the expression is (x - 2) + (5 - x) = 3, a constant.



|x - 2| + |x - 5| is constant on [2,5].

Inside the interval its derivative is zero.

There are corners at the endpoints.

$$f'(3) = 1 - 1 = 0$$

Watch for. At a corner, averaging the left and right derivatives does not produce a derivative.

Differentiation

Derivative rules, implicit and parametric forms

01 Differentiate products and quotients

$$(uv)' = u'v + uv', \quad \left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$

For a quotient, v must be nonzero.

02 Apply the chain rule

$$\frac{d}{dx} f(g(x)) = f'(g(x))g'(x)$$

The chain rule accounts for the inner derivative.

03 Differentiate parametric equations

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}, \quad \frac{d^2y}{dx^2} = \frac{d(dy/dx)/dt}{dx/dt}$$

These forms require dx/dt not zero at the point.

04 Recall basic derivative pairs

$$(\sin x)' = \cos x, \ (e^x)' = e^x, \ (\ln x)' = 1/x$$

Radian convention; derivative of arcsin x is 1/sqrt(1-x^2) on (-1,1).

Quick method. For y = u(x) to the power v(x), logarithmic differentiation can simplify the work, using a positive real base. In implicit differentiation, every y derivative brings a dy/dx.

EXAMPLE 1 **Differentiate y = ln(1 + x^2).**

Use the logarithm rule outside and the derivative 2x inside.

$$y' = \frac{2x}{1 + x^2}$$

EXAMPLE 2 **For x = t^2 and y = t^3, find dy/dx and the second derivative at t = 2.**

First divide $3t^2$ by 2t. Differentiate that result with respect to t, then divide by 2t.

$$\frac{dy}{dx} = \frac{3t}{2}; \quad \left(\frac{dy}{dx}, \frac{d^2y}{dx^2}\right)_{t=2} = \left(3, \frac{3}{8}\right)$$

Watch for. The derivative of f(g(x)) is not merely f prime of g(x). A missed inner derivative often changes the answer by a factor.

Applications of Derivatives

Tangents, monotonicity and extrema

01 Use the derivative as tangent slope

$$y - y_0 = f'(x_0)(x - x_0)$$

Tangent at a point with finite slope; normal slope is -1/f prime when nonzero.

02 Read the sign of the first derivative

$$f'(x) > 0 \Rightarrow f \text{ increasing}, \quad f'(x) < 0 \Rightarrow f \text{ decreasing}$$

Apply on an interval where the derivative exists.

03 Classify a stationary point

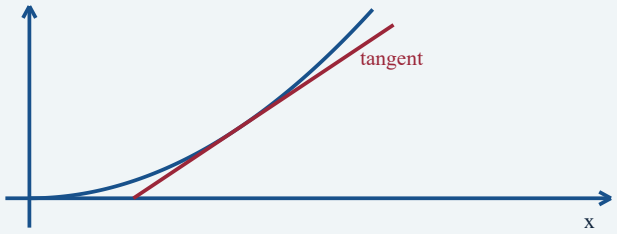
$$f'(c) = 0,\ f''(c) > 0 \Rightarrow \text{local minimum}$$

A negative second derivative gives a local maximum; zero is inconclusive.

Quick method. For an absolute maximum/minimum on a closed interval, compare endpoint values and all interior critical points. A sign-change test can work when the second derivative test fails.

EXAMPLE 1 Find the tangent to $y = x^2$ at $x = 2$.

The point is (2,4) and slope is 4. Insert them into point-slope form.



A tangent has the local derivative as its slope.

Use the point and the slope together.

A stationary tangent is horizontal.

$$y - 4 = 4(x - 2) \Rightarrow y = 4x - 4$$

EXAMPLE 2 Find the absolute minimum of $x^2 - 4x + 7$ on [0,3].

Complete the square to (x - 2) squared + 3. The vertex x = 2 lies inside the interval.

$$\min f = 3 \quad \text{at } x = 2$$

Watch for. A stationary point need not be an extremum: x^3 at zero is a standard counterexample.

Integration

Standard antiderivatives and hyperbolic functions

01 Integrate powers and the reciprocal

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C\ (n \neq -1), \qquad \int \frac{dx}{x} = \ln|x| + C$$

Use intervals where the real power is defined and differentiable; the reciprocal formula excludes zero.

02 Recognise exponential and arctangent forms

$$\int e^{ax}\,dx = \frac{e^{ax}}{a} + C, \qquad \int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$

a not zero in the exponential formula; a > 0 in the arctangent formula.

03 Define hyperbolic sine and cosine

$$\sinh x = \frac{e^x - e^{-x}}{2}, \qquad \cosh x = \frac{e^x + e^{-x}}{2}$$

cosh squared x - sinh squared x = 1.

04 Integrate hyperbolic sine and cosine

$$\int \cosh x\,dx = \sinh x + C, \qquad \int \sinh x\,dx = \cosh x + C$$

These are hyperbolic functions, not ordinary trigonometric ones.

Quick method. Differentiate your proposed antiderivative to verify it. Check the coefficient of the inner x before writing the answer.

EXAMPLE 1 **Integrate $3x^2 + 2/x$ for $x > 0$.**

Integrate the polynomial and reciprocal terms separately.

$$\int \left(3x^2 + \frac{2}{x}\right)dx = x^3 + 2\ln x + C$$

EXAMPLE 2 **Integrate cosh(2x).**

Use the reverse chain rule. The derivative of sinh(2x) is twice the integrand.

$$\int \cosh 2x\,dx = \frac{1}{2}\sinh 2x + C$$

Watch for. Every indefinite integral represents a family and needs an arbitrary constant; different intervals may have different constants.

Integration

Substitution, parts and partial fractions

01 Substitute a function with its derivative

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du$$

Choose u = g(x); transform the whole integrand and dx.

02 Integrate by parts

$$\int u\,dv = uv - \int v\,du$$

Choose u so its derivative simplifies the remaining integral.

03 Separate a rational expression into fractions

$$\frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1}$$

Partial fractions are used on intervals avoiding the poles.

Quick method. A function beside its derivative suggests substitution. A polynomial times an exponential suggests integration by parts. Divide first if a rational function is improper.

EXAMPLE 1 **Integrate $2x/(1 + x^2)$.**

Set $u = 1 + x^2$; $du = 2x\,dx$.

$$\int \frac{2x}{1+x^2}\,dx = \ln(1+x^2) + C$$

EXAMPLE 2 **Evaluate the integral of $x\,e^x$ from 0 to 1.**

By parts, take $u = x$ and $dv = e^x\,dx$. Evaluate $(x - 1)e^x$ at the endpoints.

$$\int_0^1 xe^x\,dx = [(x-1)e^x]_0^1 = 1$$

Watch for. When substituting in a definite integral, either change the bounds or return to x before using the original bounds.

Integration

Definite integrals and symmetry

01 Reflect an integral in its midpoint

$$\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$$

Reflection about the interval midpoint preserves the integral.

02 Exploit odd symmetry on a symmetric interval

$$\int_{-a}^{a} f(x)\,dx = 0 \text{ for odd } f$$

For even f, the integral is twice the integral from 0 to a; assume integrability.

03 Pair sine and cosine under reflection

$$\int_0^{\pi/2} \frac{f(\sin x)}{f(\sin x)+f(\cos x)}\,dx = \frac{\pi}{4}$$

The integrand must be defined and integrable; reflection pairs it with its complement.

Quick method. Write the reflected integral alongside the original. Adding often cancels the difficult ratio and leaves a constant integrand.

EXAMPLE 1 **Evaluate the integral from -2 to 2 of $x^3 + x^2$.**

The odd part integrates to zero. Double the integral of x^2 from 0 to 2.

$$\int_{-2}^{2}(x^3+x^2)dx = \frac{16}{3}$$

EXAMPLE 2 **Evaluate the integral from 0 to π/2 of sin x/(sin x + cos x).**

Replace x by π/2 - x. The two integrands add to 1, so twice the integral equals π/2.

$$2I = \frac{\pi}{2} \Rightarrow I = \frac{\pi}{4}$$

Watch for. Symmetry cannot repair a divergent improper integral. Check that the denominator does not create a singularity.

Integration

Area between curves

01 Integrate the absolute vertical separation

$$A = \int_a^b |f(x) - g(x)|\,dx$$

Split at crossings before removing the absolute value.

02 Use horizontal strips when simpler

$$A = \int_c^d [x_{\text{right}}(y) - x_{\text{left}}(y)]\,dy$$

Horizontal strips may avoid awkward inverse functions.

03 Distinguish a signed integral from area

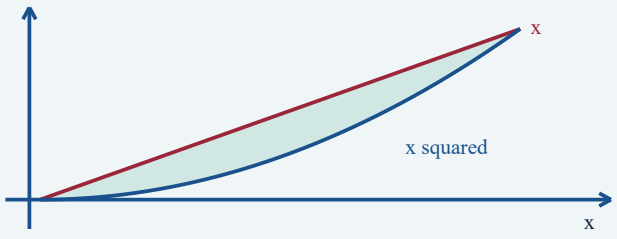
$$\int_a^b f(x)\,dx$$

This gives signed area; geometric area is nonnegative.

Quick method. Find intersections, sketch which curve is above, then choose vertical or horizontal strips. Symmetry can halve the work after the region is identified.

EXAMPLE 1 Find the area enclosed by y = x and $y = x^2$.

Intersections are x = 0 and 1. The line lies above the parabola between them.



Between 0 and 1, x is above x squared.

Area = integral of upper minus lower.

The two intersections fix the bounds.

$$A = \int_0^1 (x - x^2)dx = \frac{1}{6}$$

EXAMPLE 2 Find the area between y = x and the x-axis on [-1,1].

The signed integral is zero, but area uses |x|. Double the positive triangular area.

$$A = 2\int_0^1 x\,dx = 1$$

Watch for. Using a single top-minus-bottom expression across a crossing can subtract two positive areas instead of adding them.

Differential Equations

Differential equations: order, degree and separation

01 Separate the variables

$$\frac{dy}{dx} = f(x)g(y) \Rightarrow \int \frac{dy}{g(y)} = \int f(x)dx$$

Division assumes g(y) not zero; check constant solutions lost by dividing.

02 Reduce a homogeneous first-order equation

$$\frac{dy}{dx} = F(y/x), \quad y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

Homogeneous first-order form; work on a suitable interval with x not zero.

03 Identify the derivative order

$$\text{order} = \text{highest derivative order}$$

Degree is the highest derivative's power only after a polynomial form in derivatives is validly obtained.

Quick method. Try separating x and y before a more elaborate method. An equation containing sin(y prime) is not polynomial in derivatives and has no ordinary degree.

EXAMPLE 1 **State the order and degree of (y double prime) cubed + (y prime) squared = 0.**

The highest derivative is second order and it appears to the third power in a polynomial equation.

$$\text{order} = 2, \quad \text{degree} = 3$$

EXAMPLE 2 **Solve dy/dx = 2xy with y(0) = 3.**

For nonzero y, separate dy/y = 2x dx and integrate. The initial condition determines the constant.

$$y = 3e^{x^2}$$

Watch for. Differentiation removes arbitrary constants but may introduce extra solutions; initial conditions select the required member.

Differential Equations

Linear equations, growth and decay

01 Find an integrating factor

$$y' + P(x)y = Q(x), \quad \mu = e^{\int P(x)dx}$$

The integrating factor is defined on the interval under consideration.

02 Integrate the product with the factor

$$\mu y = \int \mu Q \, dx + C$$

Multiply the full equation by the integrating factor.

03 Model exponential growth and decay

$$N' = kN \Rightarrow N = N_0 e^{kt}, \quad t_{1/2} = \frac{\ln 2}{\lambda}$$

Decay uses k = -lambda < 0; half-life requires lambda > 0.

Quick method. Put the coefficient of y prime equal to 1 before finding the integrating factor. In growth questions, a time ratio can eliminate the initial quantity.

EXAMPLE 1 **Solve y prime + y = 2 with y(0) = 0.**

The integrating factor is e^x. Integrate (e^x y) prime = $2e^x$, then apply the condition.

$$y = 2(1 - e^{-x})$$

EXAMPLE 2 **A substance has half-life 4 hours. What fraction remains after 12 hours ?**

Twelve hours contains three half-lives, so multiply the remaining fraction by 1/2 three times.

$$\frac{N}{N_0} = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$$

Watch for. Doubling time and half-life use natural logarithms. Exponential growth is not the same as adding a fixed amount each interval.

Vector Algebra

Vector components, dot product and projection

01 Find magnitude and a unit direction

$$|\mathbf{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2}, \quad \hat{\mathbf{a}} = \frac{\mathbf{a}}{|\mathbf{a}|}$$

A unit vector in the direction of a requires a not zero.

02 Relate the dot product to angle

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$$

For nonzero vectors, zero dot product means perpendicular directions.

03 Project one vector onto another

$$\text{proj}_{\mathbf{b}}\mathbf{a} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|^2}\mathbf{b}$$

The scalar projection divides by |b| instead; b must be nonzero.

Quick method. Expand dot products in components when coordinates are given. For an angle, retain the sign unless the problem asks for the acute angle between lines.

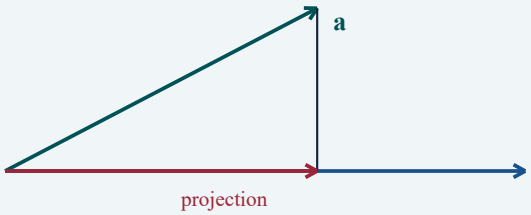
EXAMPLE 1 Find the angle between a = (1,1,0) and b = (1,0,1).

The dot product is 1 and each magnitude is square root of 2.

$$\cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$$

EXAMPLE 2 Project a = (2,3,0) onto b = (1,0,0).

The dot product is 2 and b has unit length. Multiply b by 2.



Project onto the direction of the second vector.

The scalar projection may be negative.

The vector projection includes direction.

$$\text{proj}_{\mathbf{b}}\mathbf{a} = (2, 0, 0)$$

Watch for. The dot product is a scalar, while the vector projection is a vector. Their units and denominators differ.

Vector Algebra

Cross product, areas and moments

01 Relate cross-product magnitude to angle

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\theta$$

Direction follows the right-hand rule; reversing order reverses sign.

02 Use the cross product for area

$$A_{\text{parallelogram}} = |\mathbf{a} \times \mathbf{b}|, \quad A_{\text{triangle}} = \tfrac{1}{2}|\mathbf{a} \times \mathbf{b}|$$

The vectors must represent adjacent sides from a common vertex.

03 Distinguish work from moment

$$W = \mathbf{F} \cdot \mathbf{s}, \quad \boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$$

The work formula assumes constant force; torque uses position r measured from the stated origin or pivot.

Quick method. For area, calculate the magnitude rather than the full direction if that is all the question needs. For torque, measure r from the stated origin or pivot.

EXAMPLE 1 **Find the triangle area formed by side vectors (2,0,0) and (0,3,0).**

Their cross product has magnitude 6. The triangle uses half the parallelogram area.

$$A = \tfrac{1}{2}(6) = 3$$

EXAMPLE 2 **A force (0,5,0) newtons acts at position (2,0,0) metres. Find its moment about the origin.**

Take r cross F in that order; the positive z direction follows the right-hand rule.

$$\boldsymbol{\tau} = (0, 0, 10)\ \text{N m}$$

Watch for. A cross product is not componentwise multiplication. Parallel vectors give zero cross product, including opposite directions.

Vector Algebra

Scalar triple product and coplanarity

01 Form the scalar triple product

$$[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$$

Cyclic permutations preserve the value; swapping two vectors reverses its sign.

02 Calculate spatial volumes

$$V_{\text{parallelepiped}} = |[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]|, \qquad V_{\text{tetrahedron}} = \tfrac{1}{6}|[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]|$$

The three edge vectors start at the same vertex.

03 Test coplanarity and linear dependence

$$[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = 0 \Leftrightarrow \text{linear dependence}$$

For three position-difference vectors this is the coplanarity test.

Quick method. Use row operations on the scalar-triple-product determinant. A repeated or proportional row makes the value zero immediately.

EXAMPLE 1 **Find the parallelepiped volume for (1,0,0), (0,2,0) and (0,0,3).**

The three edges are mutually perpendicular; equivalently evaluate their diagonal determinant.

$$V = |1 \cdot 2 \cdot 3| = 6$$

EXAMPLE 2 **Are a = (1,2,3), b = (2,4,6), c = (0,1,0) coplanar ?**

Since b = 2a, the three vectors are dependent and their scalar triple product is zero.

$$[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = 0 \Rightarrow \text{coplanar}$$

Watch for. A signed triple product may be negative, but a geometric volume cannot be negative.

Statistics

Data displays, mean, median and mode

01 Use frequencies for means and sectors

$$\bar{x} = \frac{\sum f_i x_i}{\sum f_i}, \qquad \theta_i = 360^\circ \frac{f_i}{N}$$

Frequency-weighted mean and pie-chart sector angle.

02 Estimate a grouped median

$$\text{Median} = L + \frac{N/2 - C_f}{f} h$$

Grouped-data estimate: L lower boundary, Cf preceding cumulative frequency, f class frequency, h class width.

03 Estimate a grouped mode

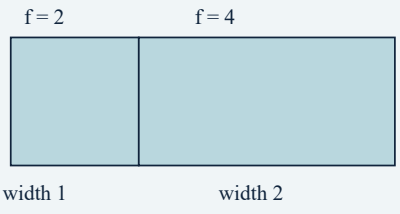
$$\text{Mode} = L + \frac{f_1 - f_0}{2f_1 - f_0 - f_2} h$$

Equal-width classes; f1 is modal frequency, f0,f2 its neighbours; denominator nonzero.

Quick method. Sort raw data before finding the median. In an unequal-class-width histogram, use frequency density so bar area represents frequency.

EXAMPLE 1 **Find the mean when values 1, 2 and 3 have frequencies 2, 3 and 5.**

Multiply each value by its frequency. The total frequency is 10 and weighted sum is 23.



With unequal class widths, plot density.

Both bars have height 2 here.

Their areas are proportional to frequencies.

$$\bar{x} = \frac{2 + 6 + 15}{10} = 2.3$$

EXAMPLE 2 **A category accounts for 25 of 100 observations. What is its pie-chart angle ?**

Use its fraction of the full 360-degree circle.

$$\theta = 360^\circ \frac{25}{100} = 90^\circ$$

Watch for. A frequency polygon joins class-midpoint frequency points; an ogive uses cumulative frequencies and class boundaries.

Statistics

Variance, standard deviation and transformations

01 Calculate variance and standard deviation

$$\sigma^2 = \frac{\sum f_i x_i^2}{N} - \bar{x}^2, \quad \sigma = \sqrt{\sigma^2}$$

Population variance divides by N; sample-variance conventions may instead use N - 1.

02 Transform location and spread

$$Y = aX + b \Rightarrow \bar{Y} = a\bar{X} + b, \quad \sigma_Y = |a|\sigma_X$$

Adding a constant changes the mean but not the spread.

03 Compare relative spread using CV

$$\text{CV} = 100\frac{\sigma}{\bar{x}}\%$$

Use for appropriate ratio-scale data with positive, nonzero means.

Quick method. For evenly spaced small data, deviations from a convenient centre reduce arithmetic. Compare coefficients of variation only when the quantities are meaningfully comparable.

EXAMPLE 1 **Find the population variance of 2, 4 and 6.**

The mean is 4. The squared deviations are 4, 0 and 4; divide by 3.

$$\sigma^2 = \frac{8}{3}, \quad \sigma = \sqrt{\frac{8}{3}}$$

EXAMPLE 2 **X has mean 5 and standard deviation 2. Find these for Y = -3X + 4.**

Transform the mean with its sign, but multiply the standard deviation by the absolute scale.

$$\bar{Y} = -11, \quad \sigma_Y = 6$$

Watch for. Variance has squared units; standard deviation has the original units. Variance cannot be negative.

Statistics

Correlation and regression

01 Normalise covariance to obtain correlation

$$r = \frac{\mathrm{Cov}(X, Y)}{\sigma_X \sigma_Y}, \qquad -1 \le r \le 1$$

Requires positive standard deviations.

02 Use the correct regression direction

$$y - \bar{y} = b_{yx}(x - \bar{x}), \qquad b_{yx} = r\frac{\sigma_y}{\sigma_x}$$

Regression of y on x predicts y from x.

03 Relate the two regression slopes

$$b_{xy} b_{yx} = r^2$$

Both regression slopes have the same sign as r; the two lines pass through (mean x, mean y).

Quick method. If both regression slopes are known, take the square root of their product and use their common sign. Keep the prediction direction clear.

EXAMPLE 1 **Regression slopes are -0.8 and -0.2. Find the correlation coefficient.**

Their product is 0.16. The shared negative sign fixes the sign of r.

$$r = -\sqrt{0.16} = -0.4$$

EXAMPLE 2 **Mean x = 2, mean y = 5, and the y-on-x slope is 3. Predict y at x = 4.**

Insert the means and slope in the regression equation.

$$y - 5 = 3(4 - 2) \Rightarrow y = 11$$

Watch for. Correlation does not establish causation. Do not interchange the x-on-y and y-on-x slopes.

Probability

Events, counting and complementary probability

01 Subtract overlap in a union

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

This identity does not require independence.

02 Use equally likely outcomes and complements

$$P(A') = 1 - P(A), \qquad P(A) = \frac{n(A)}{n(S)}$$

The counting formula requires a finite equally likely sample space.

03 Recognise disjoint events

$$P(A \cap B) = 0 \text{ for disjoint events}$$

Disjoint means cannot occur together, not independent.

Quick method. At least one is often easier as 1 minus none. Enumerate ordered outcomes when distinguishable dice or successive draws are involved.

EXAMPLE 1 **For two fair independent dice, find the probability that the sum is 8.**

The favourable ordered pairs are (2,6), (3,5), (4,4), (5,3), (6,2). There are 36 equally likely pairs.

$$P = \frac{5}{36}$$

EXAMPLE 2 **Three fair independent coins are tossed. Find the probability of at least one head.**

Use the complement: the only outcome without a head is TTT.

$$P = 1 - \left(\frac{1}{2}\right)^3 = \frac{7}{8}$$

Watch for. Equiprobable outcomes must be justified: sums of two dice are not equally likely.

Probability

Conditional probability, independence and Bayes

01 Condition on the information supplied

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \quad P(A \cap B) = P(A)P(B|A)$$

The first conditional formula needs P(B) > 0; the second uses P(B|A) and needs P(A) > 0.

02 Test independence

$$P(A \cap B) = P(A)P(B)$$

This is the independence condition; positive-probability disjoint events are not independent.

03 Reverse a conditional probability with Bayes

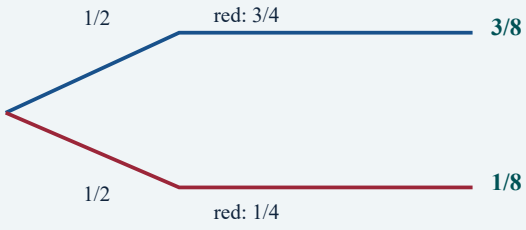
$$P(H_i|E) = \frac{P(H_i)P(E|H_i)}{\sum_j P(H_j)P(E|H_j)}$$

Hypotheses form a disjoint exhaustive partition; the evidence has positive probability.

Quick method. Without replacement, update both numerator and denominator at every draw. In Bayes problems, calculate each source's contribution to the evidence first.

EXAMPLE 1 A bag contains 3 red and 2 blue balls. Two are drawn without replacement. Find the probability both are red.

The first red has probability 3/5. After it, 2 red remain out of 4 balls.



Multiply along each branch.

Add branches leading to the observed event.

Bayes: selected branch / total evidence.

$$P = \frac{3}{5}\frac{2}{4} = \frac{3}{10}$$

EXAMPLE 2 Two equally likely bags contain red fractions 3/4 and 1/4. A red ball is drawn. What is the probability the first bag was used ?

The evidence contributions are 3/8 and 1/8. Divide the first by their total.

$$P(B_1|R) = \frac{3/8}{3/8 + 1/8} = \frac{3}{4}$$

Watch for. P(A given B) is not generally P(B given A). Reversing the condition without accounting for base rates gives wrong answers.

Probability

Random variables and binomial distribution

01 Compute expectation and variance

$$E(X) = \sum_x xp(x), \quad \operatorname{Var}(X) = E(X^2) - [E(X)]^2$$

Probabilities are nonnegative and sum to 1; the first two moments must be finite for these finite-valued formulas.

02 Recognise a binomial probability

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

Binomial model: fixed n independent trials with two outcomes and constant success probability p.

03 Use binomial mean and variance

$$E(X) = np, \quad \operatorname{Var}(X) = np(1-p)$$

These are the binomial mean and variance, not formulas for every random variable.

Quick method. Use the complement for at least one success. If mean and variance are given, their ratio gives 1 - p, then recover n.

EXAMPLE 1 **A fair coin is tossed 4 times. Find the probability of exactly two heads.**

Choose the two successful positions, then multiply the probabilities of all four tosses.

$$P = \binom{4}{2}\left(\frac{1}{2}\right)^4 = \frac{3}{8}$$

EXAMPLE 2 **A binomial variable has mean 6 and variance 3. Find n and p.**

Divide variance by mean: 1 - p = 1/2. Then n = 6/p.

$$p = \frac{1}{2}, \quad n = 12$$

Watch for. Sampling without replacement is generally not binomial because the success probability changes from draw to draw.

Exact values and trigonometric toolkit

Use exact values and radians in calculus. Every identity assumes its expressions are defined.

Standard sines

$$\sin(0, \pi/6, \pi/4, \pi/3, \pi/2) = (0, 1/2, 1/\sqrt{2}, \sqrt{3}/2, 1)$$

Standard cosines

$$\cos(0, \pi/6, \pi/4, \pi/3, \pi/2) = (1, \sqrt{3}/2, 1/\sqrt{2}, 1/2, 0)$$

Standard tangents

$$\tan(0, \pi/6, \pi/4, \pi/3) = (0, 1/\sqrt{3}, 1, \sqrt{3})$$

Cosine addition

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

Tangent addition

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

Triple cosine

$$\cos 3A = 4\cos^3 A - 3\cos A$$

Half-angle squares

$$\sin^2 \frac{A}{2} = \frac{1 - \cos A}{2}, \quad \cos^2 \frac{A}{2} = \frac{1 + \cos A}{2}$$

Cosine sum

$$\cos A + \cos B = 2\cos \frac{A + B}{2} \cos \frac{A - B}{2}$$

Sine product

$$2\sin A \sin B = \cos(A - B) - \cos(A + B)$$

Cosine equation

$$\cos x = \cos a \Rightarrow x = 2n\pi \pm a, \quad n \in \mathbb{Z}$$

Cosine rule

$$a^2 = b^2 + c^2 - 2bc\cos A$$

Triangle area

$$\Delta = \frac{1}{2}bc\sin A = rs = \frac{abc}{4R}$$

Tangent is undefined at π/2 + n*pi. Choose the sign of a half-angle square root from the angle range, not from the square identity alone.

Derivative and integral pairs

For real-valued expressions on their valid intervals. Add + C to every indefinite integral.

Power / logarithm

$$\frac{d}{dx}x^n = nx^{n-1}; \quad \int x^n dx = \frac{x^{n+1}}{n+1} + C \ (n \neq -1)$$

Reciprocal

$$\frac{d}{dx}\ln|x| = \frac{1}{x}; \quad \int \frac{dx}{x} = \ln|x| + C$$

Exponential

$$\frac{d}{dx}a^x = a^x \ln a; \quad \int a^x dx = \frac{a^x}{\ln a} + C \ (a > 0, \ a \neq 1)$$

Sine

$$(\sin x)' = \cos x; \quad \int \sin x \, dx = -\cos x + C$$

Cosine

$$(\cos x)' = -\sin x; \quad \int \cos x \, dx = \sin x + C$$

Tangent / cotangent

$$(\tan x)' = \sec^2 x, \quad (\cot x)' = -\csc^2 x$$

Secant / cosecant

$$(\sec x)' = \sec x \tan x, \quad (\csc x)' = -\csc x \cot x$$

Inverse sine / cosine

$$(\sin^{-1} x)' = \frac{1}{\sqrt{1-x^2}}, \quad (\cos^{-1} x)' = -\frac{1}{\sqrt{1-x^2}}$$

Inverse tangent

$$(\tan^{-1} x)' = \frac{1}{1+x^2}$$

Log-trig integrals

$$\int \tan x \, dx = -\ln|\cos x| + C, \quad \int \cot x \, dx = \ln|\sin x| + C$$

Quadratic-root form

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}(x/a) + C \quad (a > 0)$$

Hyperbolic pair

$$(\sinh x)' = \cosh x, \quad (\cosh x)' = \sinh x$$

Arcsine/arccosine derivatives require -1 < x < 1. In the root-form integral, |x| < a. Apply the chain rule to composite inputs; e^x is the separate a = e case.

Geometry, vectors and matrix identities

Match the standard form and the axis before substituting numbers. Nonzero-denominator conditions still apply.

2 by 2 inverse

$$\det A = ad - bc; \quad A^{-1} = \frac{1}{ad-bc}\,\mathrm{adj}(A) \quad (ad - bc \neq 0)$$

Circle tangent length

$$L = \sqrt{(x_0 - h)^2 + (y_0 - k)^2 - r^2} \quad \text{(external point)}$$

Ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1: \quad c^2 = a^2 - b^2,\ e = c/a \quad (a > b > 0)$$

Ellipse directrices

$$x = \pm a/e, \quad \text{latus rectum length} = 2b^2/a$$

Hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1: \quad c^2 = a^2 + b^2,\ e = c/a$$

Hyperbola directrices

$$x = \pm a/e, \quad \text{latus rectum length} = 2b^2/a$$

Distance: parallel planes

$$D = \frac{|d_1 - d_2|}{\sqrt{a^2 + b^2 + c^2}}$$

Planes angle

$$\cos\theta = \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{|\mathbf{n}_1||\mathbf{n}_2|}$$

Squared vector sums

$$|\mathbf{a} \pm \mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 \pm 2\mathbf{a} \cdot \mathbf{b}$$

Cross-product magnitude

$$|\mathbf{a} \times \mathbf{b}|^2 = |\mathbf{a}|^2|\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2$$

Vector triple product

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b})$$

Adjoint determinant

$$\det(\mathrm{adj}A) = (\det A)^{n-1} \quad (n \geq 2)$$

For parallel-plane distance, first give both equations the same (a,b,c) coefficients. Rotating a conic swaps its x/y roles. A circle has eccentricity 0; its finite directrix formula does not apply.

Sums, counting and final decision checks

A compact last-revision sheet. Conditions are part of each shortcut.

Natural-number sums

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}, \qquad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

Sum of cubes

$$\sum_{k=1}^{n} k^3 = \left[\frac{n(n+1)}{2}\right]^2$$

Weighted binomial sum

$$\sum_{r=0}^{n} r\binom{n}{r} = n2^{n-1} \quad (n \geq 1)$$

Three-set inclusion

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|$$

Independent repeats

$$P(\text{at least one success}) = 1 - (1-p)^n$$

Total probability

$$P(E) = \sum_{i} P(H_i)P(E|H_i)$$

Combined mean

$$\bar{x} = \frac{n_1\bar{x}_1 + n_2\bar{x}_2}{n_1 + n_2}$$

Symmetric roots

$$\alpha^3 + \beta^3 = (\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta)$$

Independent repeats assume fixed n and constant success probability p. Total probability uses a disjoint exhaustive partition with positive-probability conditioning events. Before selecting an option: check the domain, angle units, excluded values, sign, dimensions and the original equation. A numerical counterexample can disprove a universal statement; a few examples cannot prove it.

Branches, domains and standard forms

These restrictions decide whether a remembered identity can actually be used.

Inverse reciprocal functions

$$\sec^{-1} x = \cos^{-1}(1/x), \quad \csc^{-1} x = \sin^{-1}(1/x) \quad (|x| \geq 1)$$

Cotangent inverse branch

$$\cot^{-1} x \in (0, \pi), \quad \tan^{-1} x + \cot^{-1} x = \pi/2$$

Line: point-slope / intercept

$$y - y_1 = m(x - x_1), \quad x/a + y/b = 1 \quad (a, b \neq 0)$$

Line: normal form

$$x\cos\alpha + y\sin\alpha = p \quad (p \geq 0)$$

3D symmetric line

$$\frac{x - x_0}{l} = \frac{y - y_0}{m} = \frac{z - z_0}{n} \quad (lmn \neq 0)$$

Plane through a point

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0$$

Plane intercepts

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \quad (abc \neq 0)$$

Formation by eliminating C

$$y = Ce^{2x} \Rightarrow y' = 2Ce^{2x} \Rightarrow y' - 2y = 0$$

Fixed point in composition

$$f(a) = a \Rightarrow f^{\circ n}(a) = a \quad (n \geq 1)$$

Generalised mean inequality

$$A \geq G \geq H > 0 \quad \text{for positive observations}$$

The inverse secant range is [0,pi] excluding π/2; inverse cosecant uses [-π/2,π/2] excluding zero. For a line with any zero direction ratio, use vector form instead of dividing by zero. The plane normal must be nonzero.